ISRAEL JOURNAL OF MATHEMATICS, Vol. 28, Nos. 1-2, 1977

SEMIPRIME RINGS WITH FINITE LENGTH W.R.T. AN IDEMPOTENT KERNEL FUNCTOR

BY

V. K. GOEL, S. K. JAIN AND SURJEET SINGH

ABSTRACT

The purpose of this note is to show that the only idempotent kernel functor σ on Mod-R, where R is a semiprime ring and R_R has finite σ-length, is the idempotent kernel functor Z corresponding to the Goldie torsion theory.

The concept of σ-chains and σ-length of a module was introduced by Goldman [2]. For all our terminology and notations we refer to Goldman [1] and [2].

Throughout all modules are right and unital. For any R-module M we denote by $\hat{M}$ the injective hull of M and for any two R-modules $A, B, A \subset' B$ denotes that A is an essential submodule of B. Also for any idempotent kernel functor σ the set of right ideals A of the ring R such that $\sigma(R/A) = R/A$ is denoted by $\mathcal{F}_\sigma$ and is called the topology with respect to σ. For any R-submodule N of M if $\sigma(M/N) = (0)$ then we say that N is a σ-closed submodule of M. The functor Z on the category Mod-R is defined by $Z(M) = \{x \in M \mid xE = 0$, for some essential right ideal E in $R\}$, and it is, in general, a kernel functor. In case $Z(R) = (0)$ then Z is an idempotent kernel functor.

THEOREM. *Let R be a semiprime ring and σ be an idempotent kernel functor on* Mod-R. *Then the following are equivalent*:

(i) *R_R has finite σ-length.*

(ii) *$\sigma(R) = (0)$, R is a right Goldie ring, and the classical quotient ring Q coincides with the ring of quotients $Q_\sigma(R)$.*

In each case $\sigma = Z$ where Z is the idempotent kernel functor corresponding to the Goldie torsion theory.

Received August 15, 1976

PROOF. (i) $\Rightarrow$ (ii). Since R_R has finite σ-length it follows that $\sigma(R) = (0)$ ([2], prop. 1.1). Thus right ideals in the topology corresponding to σ are all essentials, showing $\sigma \leqq Z$. Further, R satisfies acc and dcc on σ-closed right ideals ([2], prop. 1.2). But it is obvious that every annihilator right ideal of R is σ-closed. Thus R satisfies acc and dcc on annihilator right ideals and so $Z(R) = (0)$. This implies R has finite Z-length ([2], cor. 1.3), proving that R is a right Goldie ring. Now R has a classical right quotient ring Q which is semisimple artinian, and $Q = \hat{R}_R$. As $\sigma(R) = (0)$, the quotient ring of R with respect to σ, namely, $Q_\sigma(R)$, is contained in Q. We proceed to show that $Q_\sigma(R) = Q$. Write $Q = \bigoplus \Sigma_{i=1}^n e_iQ$, e_iQ minimal right ideals of Q. Let $P_i = e_iQ \cap Q_\sigma(R)$. Since P_i has no essential extension in $Q_\sigma(R)$, each P_i is Z-closed and hence a σ-closed R-submodule of $Q_\sigma(R)$. Also it is easy to see that P_i is a minimal Z-closed R-submodule of $Q_\sigma(R)$. As $Q_\sigma(R)$ is σ-injective and P_i is a σ-closed R-submodule of $Q_\sigma(R)$, it follows that P_i is σ-injective. Further, $Q_\sigma(R)$ is a semiprime Goldie ring since $R \subset' Q_\sigma(R) \subset' Q$.

Now we show that P_i is a minimal right ideal of $Q_\sigma(R)$. Let A be a non-zero right ideal such that $A \subsetneqq P_i$. Then $AP_i \neq (0)$. Let $a \in A$ such that $aP_i \neq (0)$. Since P_i is a minimal Z-closed R-submodule of $Q_\sigma(R)$, we get $P_i \cong aP_i \subsetneqq P_i$. This gives an infinite properly descending chain $P_i \supsetneqq a_1P_i \supsetneqq a_1a_2P_i \supsetneqq \cdots$ of σ-injective right R-submodules and hence of σ-closed right R-submodules of $Q_\sigma(R)$ for some elements $a_1, a_2, \cdots$, in A. However, by Goldman ([2], cor. 1.6 and prop. 1.2), $Q_\sigma(R)$ also satisfies acc and dcc on σ-closed right R-submodules. So the above properly descending chain leads to a contradiction. Hence each P_i is a minimal right ideal in $Q_\sigma(R)$. Clearly, $P = \bigoplus \Sigma_{i=1}^n P_i \subset' Q_\sigma(R)$. Thus socle $(Q_\sigma(R)) = P$. This yields $Q_\sigma(R) =$ socle $(Q_\sigma(R))$ and hence $Q_\sigma(R) = Q$.

(ii) $\Rightarrow$ (i). Obvious.

We now show that under any of the equivalent conditions in the theorem, $\sigma = Z$. We only need to show $Z \leqq \sigma$ or equivalently $\mathcal{F}_Z \subset \mathcal{F}_\sigma$. Let $I \in \mathcal{F}_Z$ then $I \subset' R$ and hence $\hat{I} = \hat{R} = Q$. Recall that $Q_\sigma(I) = \{x \in \hat{I} \mid xA \subset I$ for some $A \in \mathcal{F}_\sigma\}$. We assert that $Q_\sigma(I)$ is a right Q-module. Let $x \in Q_\sigma(I)$ and $q \in Q$. $x \in Q_\sigma(I)$ implies $xA \subset I$ for some $A \in \mathcal{F}_\sigma$. Also $A \in \mathcal{F}_\sigma$ gives $Q_\sigma(A) = Q_\sigma(R)$. Since $q \in Q = Q_\sigma(R) = Q_\sigma(A)$, there exists $B \in \mathcal{F}_\sigma$ such that $qB \subset A$. Now $xqB \subset xA \subset I$, yielding $xq \in Q_\sigma(I)$ and hence $Q_\sigma(I)$ is a right Q-module as asserted. But then $Q_\sigma(I) \subset' Q$ implies $Q_\sigma(I) = Q$. Hence R/I is a submodule of $Q_\sigma(I)/I$. Therefore R/I is σ-torsion, proving that $I \in \mathcal{F}_\sigma$ as desired.

REMARK. Let R be a semiprime right Goldie ring with $\sigma(R) = 0$. Then $\sigma = Z$ if and only if R has finite σ-length.

References

1. O. Goldman, *Rings and modules of quotients*, J. Algebra **13** (1969), 10–47.
2. O. Goldman, *Elements of non-commutative arithmetic* 1, J. Algebra **35** (1975), 308–341.

Ohio University
Athens, Ohio 45701 USA

and

Guru Nanak Dev University
Amritsar, India